Propagation of stress waves in viscoelastic media

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A new method for the determination of the characteristic parameters of the stress wave propagation, such as attenuation coefficient, wave velocity, Young's modulus and the viscosity coefficient, at various frequencies in viscoelastic rods is presented. The method is based on the propagation of an arbitrary pulse in such a rod and the determination of the characteristic wave propagation parameters of each term of the Fourier series expansion of the propagating pulse. Since each term corresponds to a definite frequency, the characteristic properties of the wave propagation over a wide frequency range can be determined, by means of one test only. The analysis is based on the assumption that the viscoelastic material obeys a Kelvin–Voigt model. The stress pulse was created by means of a steel ball, projected by an air-gun, and was recorded by a transient recorder with digital memory. Two typical viscoelastic materials were tested, namely a poly(methyl methacrylate) (plexiglas) and a polycarbonate of bisphenol A (lexan). The characteristic wave propagation parameters were determined in a frequency range between 3 and 35 kHz and the results obtained agreed satisfactorily with corresponding results of previous investigators.

INTRODUCTION

A great number of papers have been devoted to the determination of the characteristic parameters of stress wave propagation, such as attenuation coefficient, wave velocity, Young's modulus and the viscosity coefficient in viscoelastic media. The determination of these quantities at various frequencies was obtained by using sinusoidal pulses usually produced by vibrators.

Nolle[1] has determined the sound velocity and the attenuation coefficient at three temperatures by using an acoustical technique. Hillier and Kolsky[2] have determined the sound velocity along filaments by using a longitudinal oscillation technique at a frequency range between 1 and 6 kHz. Ballow and Smith[3] have studied various physical properties of polymers at high, medium, and low frequencies. Protzman[4], by using an optical method, has determined the sound velocity in plexiglas as a function of temperature, in the frequency range between 3 and 11 MHz. The sound velocity for the same frequency range was also determined by Melchor and Petrauskas[5], by using a pulse-ultrasonic beam method. The same investigators have also determined the attenuation coefficient in plexiglas at frequencies of 0.5, 1.0 and 2.0 MHz.

Kolsky[6] has also studied the propagation of short mechanical pulses along rods. The pulses were produced by detonating small quantities of explosive at the one end of the rod. Kolsky employed a condenser microphone to record the displacement of the opposite end. By using the values of the dynamic elastic constants of the materials, determined from measurements with sinusoidally applied stresses over a wide frequency range, Kolsky has predicted the pulse shape, by means of a numerical Fourier synthesis. By comparing the thus derived pulse shape with its corresponding experimental form, he has concluded that both types of pulses are almost identical in shape. Also, by assuming that the damping of the material is not high and that it remains constant over a wide frequency range, Kolsky represented the pulse shape in a universal manner for all high polymers and for all travelling distances. Sutton[7] has studied the strain waves caused by cavitation and computed the wave velocity, the attenuation coefficient and Young's modulus of the respective material. Norris[8] has used a Hopkinson bar to investigate the attenuation coefficient and the propagation velocity of a stress pulse. Felix[9] determined the attenuation coefficient and the wave propagation velocity in plexiglas in the frequency range between 1 and 10 MHz. Goodbread *et al.*[10] have studied the velocity of axial waves in plexiglas in the frequency range between 100 and 300 kHz, by using a interferometric method. Hatfield[11] has studied the velocity and the attenuation coefficient in plexiglas for frequencies 350–750 kHz. Young's modulus of the same material was determined at low frequencies by Koppelman[12] and at medium frequencies in plexiglas and lexan by Theocaris and coworkers[13].

In all the above-mentioned papers it was found that the attenuation coefficient, Young's modulus and the wave propagation velocity increase with the frequency of the wave In the present paper a new method for the determination of the characteristic parameters of the stress-wave propagation at various frequencies in viscoelastic media was developed, whereby only one test is executed and all the necessary information needed for the determination of the parameters of the material is extracted when the specimen is under the same loading conditions. The method was applied to two typical viscoelastic materials, plexiglas and lexan. The results obtained compared favourably with existing results of other investigators.

DEVELOPMENT OF THE METHOD

Let us consider a one-dimensional stress wave, propagating along a viscoelastic rod, whose diameter is assumed to be small compared with its length. If the viscoelastic material is considered to behave at any frequency as a Kelvin–Voigt solid, the stress–strain relationship will be expressed by[14]:

$$\sigma = E'\epsilon + \eta \frac{d\epsilon}{dt} \tag{1}$$

where σ is the applied stress, ϵ the corresponding strain, E' Young's modulus, η the viscosity coefficient and t the time. If we consider that the strain ϵ varies harmonically with time, i.e. if:

$$\epsilon = \epsilon_0 \exp(i\omega t) \tag{2}$$

where ϵ_0 is the strain corresponding to $t = 0$ and ω is the circular frequency, then relation (1) takes the form:

$$\sigma = (E' + i\omega\eta)\epsilon \tag{3}$$

The quantity:

$$E^* = E' + i\omega\eta = E' + iE'' \tag{4}$$

connecting the stress with the strain is the complex Young's modulus and the quantities E' and E'' the storage and loss moduli respectively. Quantities E' and E'' may, therefore, assume the form:

$$E' = E^* \cos\delta$$

$$E'' = E^* \sin\delta \tag{5}$$

where the quantity:

$$\tan\delta = \frac{E''}{E'} \tag{6}$$

is the loss factor of the viscoelastic material.

The propagation of the one-dimensional stress wave in the rod is governed by the following differential equation:

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} \tag{7}$$

where u is the displacement along the axis x of the rod and ρ the density of the rod material.

By taking into account equations (3), (4) and the fact that:

$$\epsilon = \frac{\partial u}{\partial x} \tag{8}$$

equation (7) assumes the form:

$$\rho \frac{\partial^2 u}{\partial t^2} = (E' + iE'') \frac{\partial^2 u}{\partial x^2} \tag{9}$$

The solution of equation (9) can be expressed in the form:

$$u = u_0 \exp(-ax) \exp\{i\omega[t - (x/c)]\} \tag{10}$$

where a is the attenuation coefficient and c is the wave velocity. These quantities are expressed by[8]:

$$a = \frac{\omega}{c} \tan\delta \tag{11}$$

$$c = \left(\frac{|E^*|}{\rho}\right)^{1/2} \left(\cos\frac{\delta}{2}\right)^{-1} \tag{12}$$

From relations (4), (11) and (12) it is concluded that the viscosity coefficient η is given by:

$$\eta = \frac{2acE'}{\omega^2 - a^2c^2} \tag{13}$$

From relations (11) to (13) it is shown that, if the quantities a and c at a given frequency are known, we can compute Young's modulus and the viscosity coefficient at this frequency.

Let a periodic pulse $f(t)$, imposed on the viscoelastic rod, be considered. By expanding the function $f(t)$ into a Fourier series we obtain:

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos kt + B_k \sin kt) \tag{14}$$

where

$$A_0 = \frac{2}{\pi} \int_0^{\pi} f(t)dt$$

$$A_k = \frac{2}{\pi} \int_0^{\pi} f(t) \cos kt dt \quad (k = 0, 1, 2, \dots)$$

$$B_k = \frac{2}{\pi} \int_0^{\pi} f(t) \sin kt dt \tag{15}$$

Equation (14) may assume the form:

$$f(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(kt - \varphi_k) \tag{16}$$

where

$$C_0 = \frac{A_0}{2}, \quad C_k = (A_k^2 + B_k^2)^{1/2}, \quad \tan\varphi_k = \frac{B_k}{A_k} \tag{17}$$

By taking the first m terms of equation (16) we obtain:

$$f(t) = C_0 + \sum_{k=1}^{m} C_k \cos(kt - \varphi_k) \tag{18}$$

The stress pulse $f(t)$, applied to the rod, can be recorded in the memory of an oscillograph, and, consequently all the above coefficients C_k and φ_k can be determined. Now, let two different pulses be considered for example the original compressive pulse and the returning tensile pulse, after reflection of the previous one at the free end of the rod, and let C_k and C_k' be the Fourier coefficients of the respective harmonics of the same order in these pulses. Then, if x is the distance travelled between the two pulses, which, for subsequent pulses and for a strain detector bonded at the mid-point of the rod, is equal to twice the half-length i.e. the length of the rod, it is valid that:

$$\frac{C_k'}{C_k} = \exp(-ax) \tag{19}$$

Also, we have:

$$\frac{c_n}{c_0} = 1 + \frac{\Delta\phi}{2\pi} \tag{20}$$

where c_n and c_0 are the velocities of the nth harmonic and the original pulse respectively and $\Delta\phi$ the phase difference between the nth harmonics of the two pulses considered.

As the quantities C_k, C_k' and x, $\Delta\phi$ and c_0 can be determined experimentally, relations (19) and (20) enable the determination of the attenuation factor a and the velocity c_n at the frequency of the corresponding harmonic. Since in the Fourier analysis of the pulses $f(t)$ and $f'(t)$, a large number of harmonics is contained, the quantities a and c_n at all corresponding frequencies can be determined. By knowing a and c_n, relations (11) to (13) enable the calculation of Young's modulus and of the viscosity coefficient. Thus, all these quantities can be determined over a wide

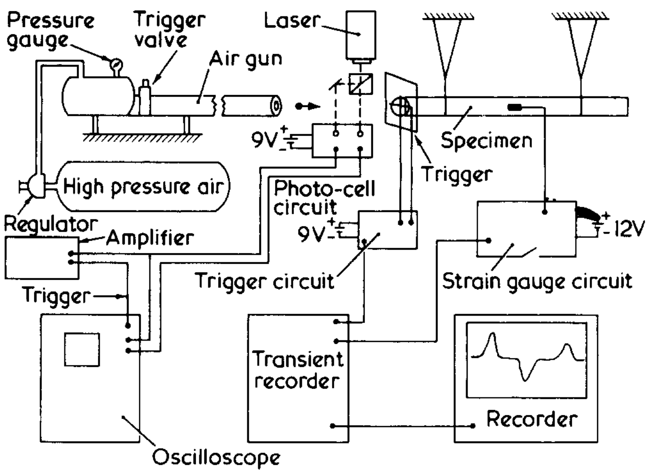

Figure 1 Diagram of the experimental arrangement

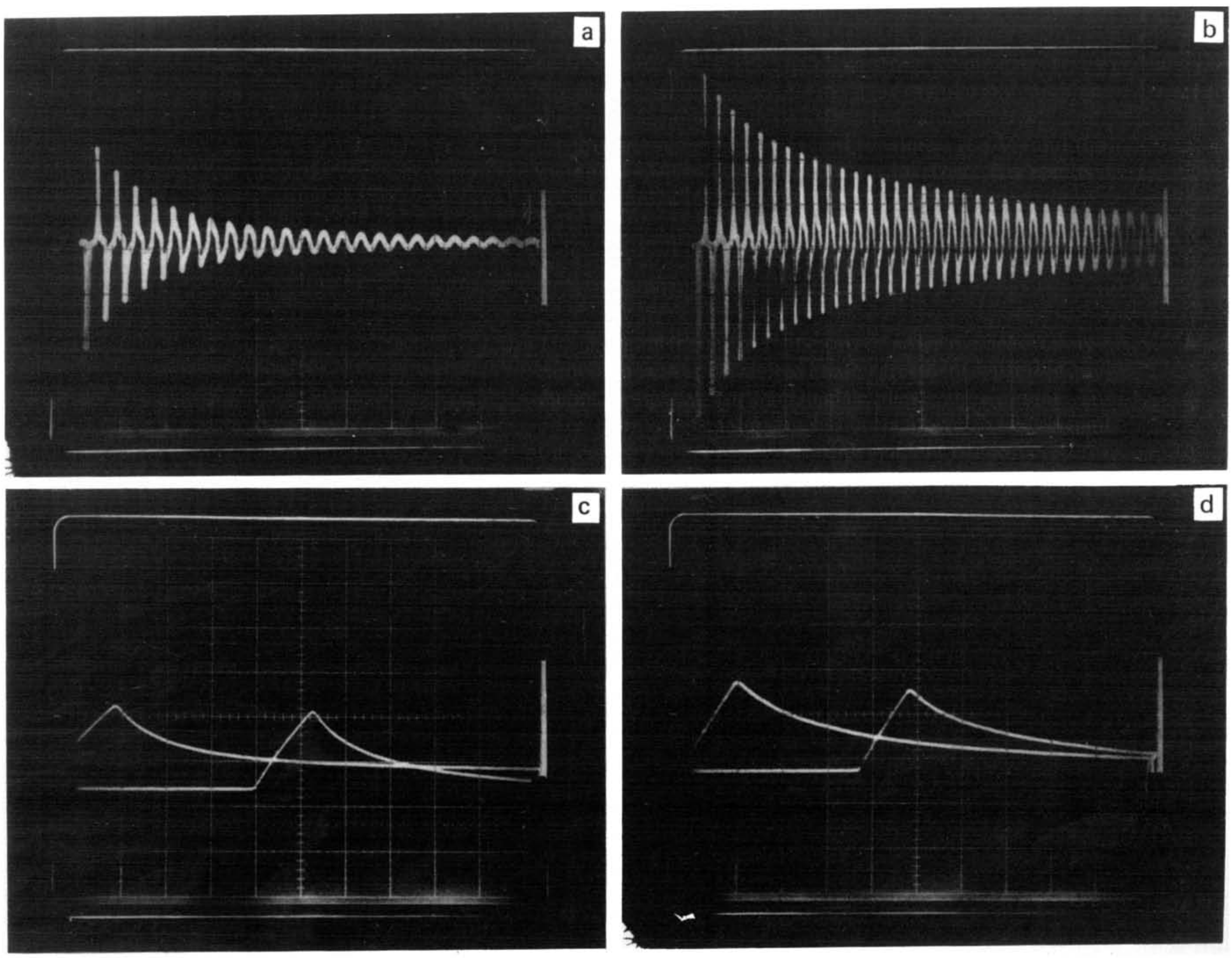

Figure 2 Typical oscillographs of the applied stress pulses for plexiglas (a) and lexan (b) with ϵ_{max} = 0.45 and 0.58%, respectively. Photocell output corresponding to sphere velocities 18 m/sec (c) and 38 m/sec (d)

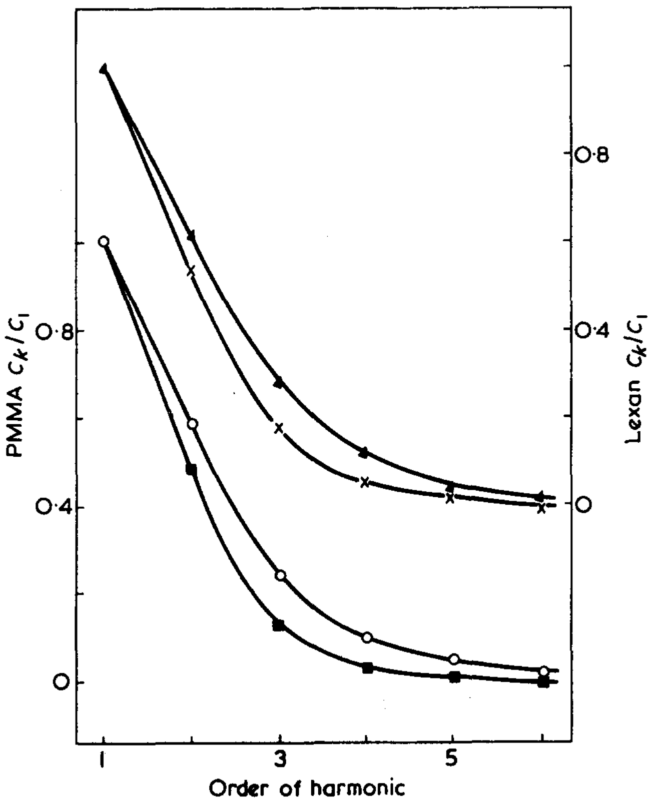

Figure 3 Variation of the ratio of the coefficients of the various harmonics over the coefficient of the fundamental harmonic for the first and sixth pulse and for plexiglas (○, ■) and lexan (▲, X)

frequency range, by using only two different arbitrary pulses, corresponding to different harmonics of the same impact.

EXPERIMENTAL

Plexiglas and lexan rods of a cross-section 1.0×1.0 cm^2 and 45 cm long were employed. The specimens were suspended at a horizontal position by means of thin strings and were impacted axially by a steel sphere of 1 cm diameter, projected from an air-gun (*Figure 1*). Three sphere velocities, selected by means of the pressure regulator of the air-gun, were used, namely 18.0, 22.0 and 38.0 m/sec. The air-gun was appropriately aligned to ensure an accurately axial impact of the specimen and it was triggered by a quick-action electromagnetic valve, so that the acceleration of the sphere in the gun was always constant.

The sphere velocity was accurately determined by measuring the time required by the sphere to run a distance of $d = 30$ mm between two parallel laser light beams, obtained from a gas laser by means of a beam splitter. A photocell with the respective circuitry was attached to either beam, and, as the beams were interrupted by the passing sphere, two pulses were fed into a storage oscilloscope. The exact time distance between the two pulses could then be accurately measured. The first pulse also triggered the oscilloscope.

The recording of the strain, produced by the stress pulse, was obtained by means of two strain gauges, 3.5 mm long, bonded at the mid-point of the rod on opposite sides, in order that the effect of flexure be counterbalanced. The gauge-length was much smaller than the length of the stress pulse, thus permitting a reliable recording of the impulse load. In order to avoid local heating of the strain gauges, due to the prolonged effect of the 50 mA electric current passing through them, and a potential destruction of the cement used, the voltage was applied only during the measurements through a special switch.

The output of the strain-gauge circuitry was recorded on a transient recorder with digital memory, which was triggered at the moment of the impact.

The oscilloscope was triggered by means of two wires, placed in front of the specimen, brought into contact by the impacting sphere. The transient recorder stored the strain wave-form in digital form and fed it at normal speed into a strip-chart recorder, through a digital-to-analogue converter. By the impact, a compressive pulse was created in the specimen, which, reflected at the free end, was returning as a tensile pulse. This tensile pulse was again reflected at the opposite end of the rod as a compressive pulse etc. These subsequent compressive and tensile pulses can be assumed as one-dimensional since the cross-section of the rod was taken to be small in comparison to the lengths of the pulses.

Figures 2a and *2b* present typical stress-pulse recordings for a maximum strain $\epsilon = 0.45$ and 0.58% respectively. *Figure 2a* refers to a plexiglas rod and for a sphere velocity of 18 m/sec, while *Figure 2b* refers to a lexan rod and for a sphere velocity of 38 m/sec.

RESULTS

The pulse recordings of *Figures 2a* and *2b* were analysed by a Fourier-series expansion and the coefficients C_k of the series were calculated. *Figure 3* presents the variation of the ratio C_k/C_1 of the coefficients of the first six harmonics in the Fourier analysis of the applied pulse for plexiglas and lexan, respectively. The frequencies of the basic harmonics were found from the corresponding frequencies of the pulse propagation and they were equal to 4.8 kHz for plexiglas and 3.3 kHz for lexan, respectively. The velocity of the pulse is calculated by measuring the time required for the pulse to propagate along the rod. This velocity was found equal to 2170 and 1485 m/sec for plexiglas and lexan, respectively.

From relations (19) and (20) the attenuation coefficient a and the wave propagation velocity c at each particular frequency, corresponding to the terms of the Fourier series of the pulse, were calculated. *Figure 4* presents the variation of these quantities, plotted against frequency, for plexiglas and lexan. By knowing the quantities a and c and by using relations (11) through (13), Young's modulus $|E^*|$ and the viscosity coefficient η were calculated. The variations of $|E^*|$ and η plotted against frequency for plexiglas and lexan, are presented in *Figure 5*.

Finally, *Figure 6* presents the variation of the attenuation coefficient and Young's modulus with frequency in the range between 10^{-3} and 10^5 kHz for plexiglas, as these quantities were given in refs 4, 5, 9, 11–13. In the same Figure, the results of the present investigation, corresponding to a frequency range between 3 and 35 kHz are also plotted. It can be seen that these results compare favourably with the results of other investigators.

CONCLUSIONS

In the present investigation an experimental method was developed for the determination of the characteristic quan-

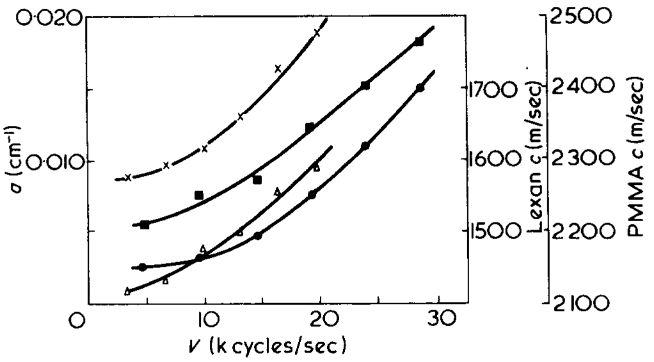

Figure 4 Variation of the attenuation coefficient and the wave propagation velocity with frequency for plexiglas (●, ■) and lexan (▲, X)

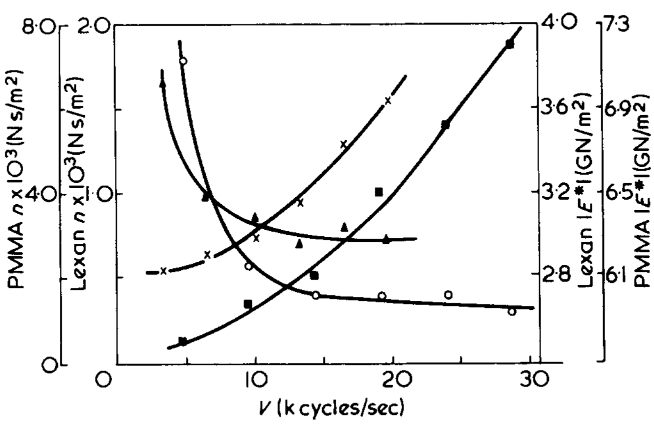

Figure 5 Variation of Young's modulus and viscosity coefficient with frequency for plexiglas (■, ○) and lexan (X, ▲)

tities of the wave propagation in Hopkinson's bars made of viscoelastic materials. The method used information from two arbitrary harmonics from the series of wave pulses to which it was analysed an arbitrary impact applied to the bar. The method constitutes the inverse idea of that developed by Kolsky[6] for the study of viscoelastic media to impact. Indeed, while Kolsky has determined the dynamic properties of polymers over a wide frequency range, by using stress pulses with different frequencies, and used Fourier synthesis to predict the shape of the composite pulse derived by this series of harmonics, in the present paper only one initial impact, i.e. one test, was used, which was further expanded to a Fourier series for the calculation of the dynamic properties of the material at all harmonics contained in the applied pulse, which can be accurately measured.

The coefficients of the Fourier series of the applied stress pulse were determined. By this method, the characteristic quantities of the wave propagation at the frequencies corresponding to each term of the Fourier expansion were determined.

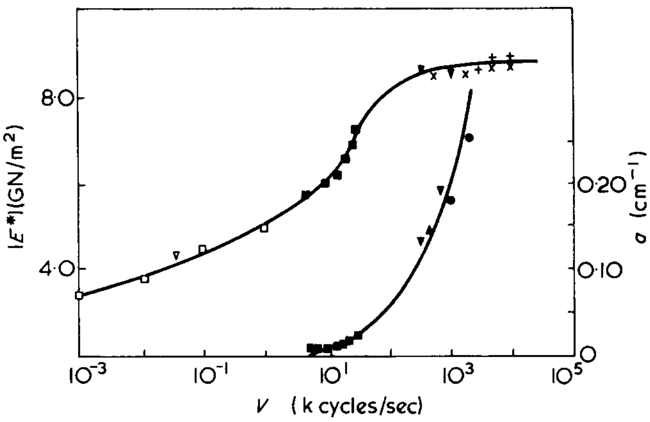

Figure 6 Variation of the attenuation coefficient and Young's modulus for plexiglas with frequency in the range 10^{-3}–10^{5} kHz given by various authors and the present work: ▼, Hatfied; ▲, Melchor and Petrauskas; ●, Felix; □, Koppelman; +, Protzman; ▽, Raftopoulos *et al.*; ■, present work

The method was applied to two typical viscoelastic materials, namely plexiglas and lexan, and the propagation velocity c, the attenuation coefficient a, the Young modulus $|E^*|$ and the viscosity coefficient η were determined as functions of frequency in the frequency range between 3 and 35 kHz. It was concluded that the quantities a, c and $|E^*|$ increase with frequency, while the viscosity coefficient η decreases. It was also deduced that for the frequency range between 3 and 35 kHz the loss factor $\tan\delta$ was practically independent of frequency and equal to 0.03 and 0.02 for plexiglas and lexan, respectively.

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